Math 055 Exam 2
Spring 2026

For full credit: Please show work using techniques from this course and use correct mathematical notation.

1. The function $y_1 = x^2$ is a solution of the differential equation $x^2y'' - 3xy' + 4y = 0$. Use reduction of order to find a second solution y_2.

Standard form: $y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0$

$P(x) = -\frac{3}{x} \Rightarrow -\int P(x)dx = \int \frac{3}{x}dx = \ln x^3$

$y_2 = y_1 \int \frac{e^{\ln x^3}}{x^4} dx = y_1 \int \frac{1}{x} dx$

$\underline{y_2 = x^2 \ln x}$

<u>OR</u> $y_2 = ux^2 \Rightarrow y_2' = u'x^2 + 2ux$

and $y_2'' = u''x^2 + 4u'x + 2u$

$x^2(u''x^2 + 4u'x + 2u) - 3x(u'x^2 + 2ux) + 4ux^2 = 0$

$u''x^4 + u'x^3 = 0$ Let $w = u'$

$w' + \frac{1}{x}w = 0$ so $\mu = e^{\int \frac{1}{x}dx} = x$

$\frac{d}{dx}(xw) = 0 \Rightarrow xw = c \Rightarrow w = \frac{c}{x}$

$u = c\ln x$ so $\underline{y_p = x^2 \ln x}$

For #2-6, solve each differential equation.

2. $y''' - y'' + y' - y = 0.$

$$m^3 - m^2 + m - 1 = 0$$

$$m^2(m-1) + 1(m-1) = 0$$

$$(m^2+1)(m-1) = 0$$

$$m = \pm i, 1$$

$$y = c_1 e^x + c_2 \cos x + c_3 \sin x$$

3. $y'' - 4y = 4x^2 + 6$

$$m^2 - 4 = 0 \Rightarrow m = \pm 2$$

$$y_c = c_1 e^{2x} + c_2 e^{-2x}$$

$$y_p = Ax^2 + Bx + C \Rightarrow y_p' = 2Ax + B$$

$$\Rightarrow y'' = 2A$$

$$2A - 4Ax^2 - 4Bx - 4C = 4x^2 + 6$$

$$A = -1 \qquad B = 0 \qquad -2 - 4C = 6$$

$$C = -2$$

$$y = c_1 e^{2x} + c_2 e^{-2x} - x^2 - 2$$

OR $\quad y = c_1 \cosh 2x + c_2 \sinh 2x - x^2 - 2$

4. $y'' - 2y' - 3y = 4e^{-x}$

$$m^2 - 2m - 3 = 0 \Rightarrow (m-3)(m+1) = 0$$

$$m = -1, 3$$

$$y_c = c_1 e^{-x} + c_2 e^{3x}$$

$y_p = Ae^{-x}$ is redundant with $4e^{-x}$, so

$$y_p = Axe^{-x} \Rightarrow y_p' = A(1-x)e^{-x}$$

$$y_p'' = A(-2+x)e^{-x}$$

$$-2A + \underline{Ax} - 2A + \underline{2Ax} - \underline{3Ax} = 4$$

$$-4A = 4 \Rightarrow A = -1$$

$$\underline{y = c_1 e^{-x} + c_2 e^{3x} - xe^{-x}}$$

5. $y'' - y' - 2y = 5\sin 2x$

$$m^2 - m - 2 = 0 \Rightarrow (m-2)(m+1) = 0$$

$$y_c = c_1 e^{2x} + c_2 e^{-x}$$

$$y_p = A\sin 2x + B\cos 2x$$

$$y_p' = 2A\cos 2x - 2B\sin 2x$$

$$y_p'' = -4A\sin 2x - 4B\cos 2x$$

$$-4A\sin 2x - 4B\cos 2x - 2A\cos 2x + 2B\sin 2x$$
$$-2A\sin 2x - 2B\cos 2x = 5\sin 2x$$

$\sin 2x$: $-4A + 2B - 2A = 5 \Rightarrow -6A + 2B = 5$

$\cos 2x$: $-4B - 2A - 2B = 0 \Rightarrow -2A - 6B = 0$

$$\begin{cases} -6A + 2B = 5 \\ 6A + 18B = 0 \end{cases} \longrightarrow 20B = 5 \Rightarrow B = \frac{1}{4}$$

Then $-6A + 2\left(\frac{1}{4}\right) = 5 \Rightarrow A = -\frac{3}{4}$

$$y = c_1 e^{2x} + c_2 e^{-x} - \frac{3}{4}\sin 2x + \frac{1}{4}\cos 2x$$

6. $x^2y'' - xy' + 5y = 0$

$$y = x^m, \quad y' = mx^{m-1}, \quad y'' = m(m-1)x^{m-2}$$

$$m(m-1) - m + 5 = 0$$

$$m^2 - 2m + 5 = 0$$

$$(m-1)^2 = \pm 2i$$

$$m = 1 \pm 2i$$

$$y = x\left[c_1 \cos(2\ln x) + c_2 \sin(2\ln x)\right]$$

7. Consider the differential equation $y'' + 2y' + y = \frac{e^{-x}}{x^2}$. The complementary function is $y_c = c_1 e^{-x} + c_2 x e^{-x}$. Use variation of parameters to find the particular solution to the differential equation.

$$W = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & (1-x)e^{-x} \end{vmatrix} = e^{-2x} - xe^{-2x} + xe^{-2x}$$

$$= e^{-2x}$$

$$W_1 = \begin{vmatrix} 0 & xe^{-x} \\ \frac{e^{-x}}{x^2} & (1-x)e^{-x} \end{vmatrix} = -\frac{e^{-2x}}{x}$$

$$W_2 = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & \frac{e^{-x}}{x^2} \end{vmatrix} = \frac{e^{-2x}}{x^2}$$

$$u_1 = \int \frac{W_1}{W}\,dx = \int -\frac{1}{x}\,dx = -\ln x = \ln\frac{1}{x}$$

$$u_2 = \int \frac{W_2}{W}\,dx = \int \frac{1}{x^2}\,dx = -\frac{1}{x}$$

$$y_p = y_1 u_1 + y_2 u_2 = e^{-x}\ln\frac{1}{x} - e^{-x}$$

8. Consider the differential equation $y'' + y = 0$. For each set of boundary conditions, find the solution to the boundary-value problem or show that no solution exists.

$$m^2 + 1 = 0 \Rightarrow m = \pm i$$

$$y = c_1 \cos x + c_2 \sin x$$

(This was worth 1 point.)

a. $y(0) = 1, y\left(\frac{\pi}{2}\right) = 3$

$y(0) = 1 \Rightarrow c_1 = 1 \quad y\left(\frac{\pi}{2}\right) = 3 \Rightarrow c_2 = 3$

$$y = \cos x + 3 \sin x$$

b. $y(0) = 0, y(\pi) = 1$

$y(0) = 0 \Rightarrow c_1 = 0 \quad y(\pi) = 1 \Rightarrow c_1 = 1$

this has no solution.

c. $y(0) = 0, y(\pi) = 0$

$y(0) = 0 \Rightarrow c_1 = 0 \quad y(\pi) = 0 \Rightarrow c_1 = 0$

There are infinitely many solutions of the form $y = c_2 \sin x$.

9. A mass weighing 8 pounds stretches a spring 2 feet. The mass is released from the equilibrium position with a downward velocity of 4 ft/s. Assuming no damping, find the position $x(t)$ of the mass at time t. Determine the amplitude, period, frequency, and phase of the motion.

$$8 = 2k \Rightarrow k = 4$$

$$Ma = 8 \Rightarrow m = \frac{8}{a} \Rightarrow m = \frac{8}{32} = \frac{1}{4}$$

$$\frac{k}{m} = 16$$

$$mx'' + kx = 0 \Rightarrow x'' + \frac{k}{m}x = 0$$

$$x(t) = c_1 \cos 4t + c_2 \sin 4t$$

$$x(0) = 0 \Rightarrow c_1 = 0$$

$x'(0) = 4$: $\quad x'(t) = 4c_2 \cos 4t$

$$4 = 4c_2 \Rightarrow c_2 = 1$$

$$x(t) = \sin 4t$$

Amp: 1 ft, freq: $\frac{2}{\pi}$ cycles/sec

Per: $\frac{2\pi}{4} = \frac{\pi}{2}$ s, Phase: 0 ft